18.5 Applications to liquid helium

- The phase diagram of helium

*Figure 18.7* Phase diagram of $^4$He. The two liquid phases are labeled I and II.
• Above the critical temperature 5.25 K, helium cannot exist as a liquid
• For 2.18 K < T < 5.25 K, compressing $^4$He isothermally causes the gas condenses to a liquid phase I
• For T < 2.18 K, isothermally compressing $^4$He forms liquid phases II
• Liquid I and II can coexist over a range of T and P, defined by the lambda line.
• Solid helium cannot exist at pressures below 25 atm, and cannot exist in equilibrium with its vapor at any T and P.
• Helium has two triple points:
  solid + liquid 1 + liquid 2
  gas + liquid 1 + liquid 2
- The heat capacity versus temperature for the two phases has the general shape of the Greek letter Λ.

- The viscosity of liquid II is virtually zero, i.e. super fluid.

- For temperature below the Λ point, liquid II is a mixture of a normal fluid having viscosity and a super fluid that has no viscosity.
- Liquid II can be regarded as Boson gas due to its weak interactions between the atoms.
- Super fluid properties of liquid \(^4\)He II could be attributed to a Bose-Einstein like condensation at the \(\Lambda\) point.

- Example: (18.15) Hydrogen freezes at 14K and boils at 20K at atmospheric pressure. The density of liquid hydrogen is 70 kg m\(^{-3}\). Hydrogen molecules are bosons. No evidence has been found for Bose-Einstein condensation in hydrogen. How do you account for this?

Solution: To have B-E condensation process take place, the temperature must reach well blow the \(T_B\).

If \(T_B\) is well below 14 K. Hydrogen gas will freeze before it can undergo B-E condensation

\[
T_B = \frac{h^2}{2\pi n k} \left( \frac{N}{2.612V} \right)^{2/3} \quad (18.38)
\]
Chapter 19. Fermi-Dirac Gases

- 19.1 The Fermi energy

1. Fermi-Dirac statistics governs the behavior of indistinguishable particles of half-integer spin (i.e. fermions).

2. It obeys Pauli exclusion principle, i.e. no more than one particle per quantum state.

- F-D distribution is

\[
f_j = \frac{N_j}{g_j} = \frac{1}{e^{(e-u)/kT} + 1}
\]
In a continuum approximation:

\[ f(\varepsilon) = \frac{N(\varepsilon)}{g(\varepsilon)} = \frac{1}{e^{(\varepsilon-u)/kT} + 1} \]

the above equations gives the probability that a single particle state \( \varepsilon \) will be occupied by a fermion.

- When \( \varepsilon = u \), \( f(\varepsilon) = \frac{1}{2} \) at any temperature.

- The chemical potential \( u \) at the temperature 0, denoted with \( u(0) \), is called the Fermi energy, written as \( \varepsilon_F \).
At Temperature $= 0$

$$(\varepsilon - \mu(0))/kT = \begin{cases} \infty & \text{if } \varepsilon < \mu(0) \\ -\infty & \text{if } \varepsilon > \mu(0) \end{cases}$$

Correspondingly

$$f(\varepsilon) = \begin{cases} 1 & \text{if } \varepsilon < \mu(0) \\ 0 & \text{if } \varepsilon > \mu(0) \end{cases}$$

- The above equation tells that at $T = 0$, all states with energy smaller than the chemical potential are occupied and all states with the energy above the chemical potential are unoccupied.
- All $N$ particles (fermions) will be crowded into the $N$ lowest energy levels, i.e. the total thermodynamic probability is 1.
- The vanishing of entropy at $T = 0$ is consistent with the third law of thermodynamics.
How does the Fermi energy \( u(0) \) depends on \( m, N \) and \( V \)?

- For particles of spin \( \frac{1}{2} \). Such as electrons, the spin factor is 2, thus

\[
g(\varepsilon)d\varepsilon = \gamma_s \frac{4\sqrt{2}\pi V}{\hbar^3} m^{3/2} \varepsilon^{1/2} d\varepsilon \quad \gamma_s = 2
\]

- For conservation of particles, \( \Sigma N_j = N \), or
\[ N = \int_0^\infty 4\pi V \left( \frac{2m}{h^2} \right)^{3/2} e^{\frac{\varepsilon}{kT}} d\varepsilon \]

Since no particle for \( \varepsilon > \mu(0) \)

\[ N = 4\pi V \left( \frac{2m}{h^2} \right)^{3/2} \int_0^{\mu(0)} \varepsilon^{3/2} d\varepsilon \]

\[ N = \frac{8\pi V}{3} \left( \frac{2m}{h^2} \right)^{3/2} \mu(0)^{3/2} \]

\[ \mu(0) = \frac{h^2}{2m} \left( \frac{3N}{8\pi V} \right)^{2/3} \]

The above eqn tells us the relationship between \( \mu(0) \) and \( N \) and \( V \)

For convenience, introduce Fermi Temperature \( T_F \) such that \( \mu(0) = \varepsilon_F = kT_F \)

\[ T_F = \frac{h^2}{2mk} \left( \frac{3N}{8\pi V} \right)^{2/3} \quad \text{or} \quad \frac{h^2}{2\pi mk} \left( \frac{N}{1.504V} \right)^{2/3} \]

analogous to the Bose temperature given in eqn. 1838

9.2 The calculation of \( \mu(T) \)

\[ N = \int_0^{\infty} f(\varepsilon) g(\varepsilon) d\varepsilon = 4\pi V \left( \frac{2m}{h^2} \right)^{3/2} \int_0^{\infty} \frac{\varepsilon^{3/2} d\varepsilon}{e^{(\varepsilon - \mu(0))/kT} + 1} \]
For $T \ll T_F$

$$M = M(0) \left[ 1 - \frac{\pi^2}{12} \left( \frac{T}{T_F} \right)^2 \right]$$

$M$ is positive for $T < T_F$, becomes negative for higher temperature.

$$f(0) = \frac{1}{e^{\frac{\mu - m/\sqrt{kT}}{kT}} + 1} < \frac{1}{2}$$

$$\frac{M}{kT} < 0 \quad \rightarrow \quad M < 0$$

At very high temperature, $T \gg T_F$

$$M = -kT \ln \left( \frac{2}{\sqrt{N}} \right)$$

With

$$\frac{8}{N} = 2 \left( \frac{2\pi m kT}{\hbar^2} \right)^{3/2} \cdot \frac{V}{N}$$
Example:

\[ f(\varepsilon) = \frac{1}{e^{\varepsilon - \mu}/kT + 1} \]

At \( T = 3T_F \)

\[ \mu = -5.6 \varepsilon_F = -5.6 kT_F \]

(a) \( \varepsilon / \varepsilon_F = 0 \) \quad \varepsilon = 0

\[ f(\varepsilon) = \frac{1}{e^{0 - (-5.6kT_F)}/kT + 1} = 0.134 \]

where \( T = 3T_F \)

(b) \( \varepsilon / \varepsilon_F = 0.5 \) \quad \varepsilon = 0.5 \varepsilon_F = 0.5 kT_F

\[ f(\varepsilon) = \frac{1}{e^{5.6 + 0.5}/kT + 1} = 0.114 \]

(c) \( \varepsilon / \varepsilon_F = 1.0 \) \quad \varepsilon = kT_F

\[ f(\varepsilon) = \frac{1}{e^{5(1.0 + 0.5)/3} + 1} = 0.0997 \]
19.3 Free electrons in a metal

1) Electrons are fermions of \( \frac{1}{2} \) spin

2) On the surface of metal, there is a strong potential barrier called the work function.

3) Work function draws back any electron that tends to make a small excursion outside.

4) The effect of electric field inside the metal is canceled out.

5) These free electrons inside the metal occupy energy states up to the Fermi-level.

6) The work function, \( \phi \), is the energy required to remove an electron at the Fermi-level from the surface.

7) In most metals, the Fermi-level of the free electrons at room temperature is only fractionally less than the Fermi-energy, \( E_F \).
Considering the influence of positive ions, the potential profile inside the metal looks like

\[
\begin{array}{c}
\text{The periodicity leads to a band structure in the density of quantum states!}
\end{array}
\]

Example: Silver, which contains one free electron per atom. The density of silver is \(10.5 \times 10^3 \text{ kg m}^{-3}\) and its atomic weight is 107.

\[
\frac{N}{V} = \frac{10.5 \times 10^3}{107} \times 6.02 \times 10^26 = 5.9 \times 10^{28} \text{ m}^{-3}
\]

The above value also represents the concentration of free electron!

For Fermi-energy

\[
\varepsilon_F = \frac{\hbar^2}{2m} \left( \frac{3N}{8\pi V} \right)^{2/3}
\]

thus

\[
m = 9.11 \times 10^{-31} \text{ kg}
\]
\[ E_F = \frac{(6.63 \times 10^{-34})^2}{2 \times 9.11 \times 10^{-31}} \left( \frac{3 \times 5.86 \times 10^{28}}{8 \pi} \right)^{2/3} \]

\[ = 8.85 \times 10^{-19} J \]

or \[ = 5.6 \text{ eV} \]

The Fermi - Temperature

\[ T_F = \frac{E_F}{k} = \frac{5.6 \text{ eV}}{8.62 \times 10^{-5} \text{ eV K}^{-1}} = 65,000 \text{ K} \]

The ratio \[ \frac{T}{T_F} = \frac{300}{6.5 \times 10^4} = 0.0046 \]

Therefore, at room temperature \( T \ll T_F \), the FHH result derived for that temperature range can be employed, such as

\[ M(T) = M(0) \left[ 1 - \frac{\pi^2}{12} \left( \frac{T}{T_F} \right)^2 \right] \]

\[ = M(0) \left[ 1 - \frac{\pi^2}{12} \times 0.0046^2 \right] \]

\[ = 0.999 M(0) \]

This result confirms that \( M(T) \) is very close to \( E_F \)

The work function, \( \phi \), depends on the metal and the condition of its surface, and is typically at the order of \( 3 - 4 \text{ eV} \).